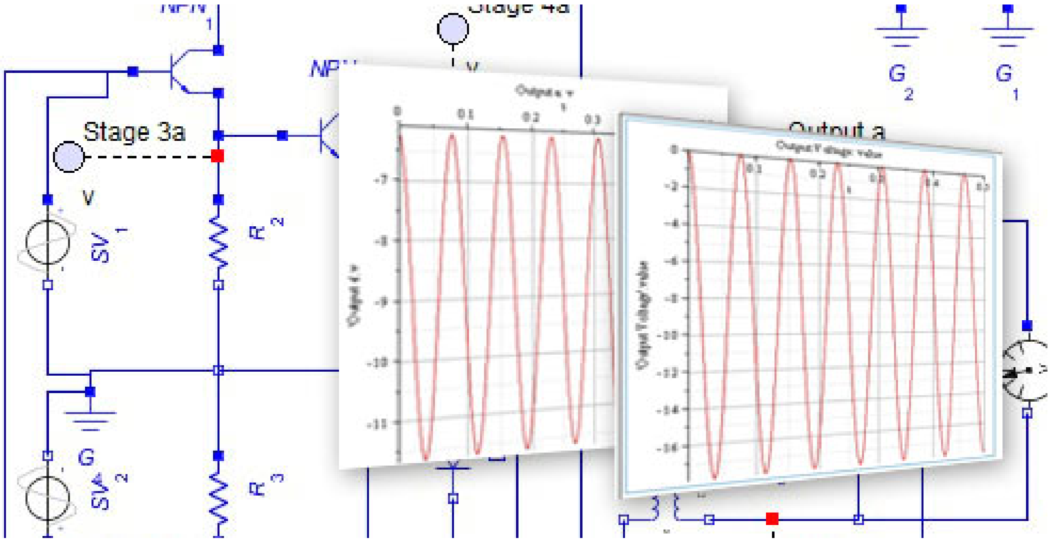
AC Power



Power generated from alternating current is an important topic to study. Different circuit elements such as capacitors and inductors each influence power delivered to a load. It's important to examine their impact on the circuit to safely and properly work with AC electricity. Power contributes to the heat produced by electronics as well as power consumption. Electric companies will save money by monitoring power lines and compensating for wasted energy. How much energy is wasted can be found with the information presented in this module.

This module explores the various forms of power within AC circuits. It looks into the power factor and provides visual aids to improve understanding. This module also looks into finding the maximum power that can be delivered to a load in a circuit.

Forms of Power

There are many forms of AC power that will be considered. This form of electricity is periodic and this will effect how we measure power consumption in AC devices. Instead of looking only at instantaneous power like what was done in DC circuits, AC circuits will react differently and this means that power needs to be treated differently to accurately measure it.

Instantaneous Power

The most basic form of power is instantaneous power. This is the measure of power flowing through a circuit element at a given moment. The instantaneous power delivered to a circuit is the product of the voltage multiplied by the velocity

$$p(t) = v(t) \cdot i(t)$$

... Eq. (1)

Here, instantaneous power is a function of time because it depends on the state of the circuit to determine it's output. Instantaneous power is often represented by the small letter *p* as opposed to the capital *P* used for average power and is measured in watts.

Average Power

This is the measure of average power being drawn through a circuit element over a full period. Average power can be found as the integral of voltage multiplied by current over one period. Voltage $v(t)$ and current $i(t)$ are periodic in an AC circuit and therefore can also be represented by $v(t+T)$ and $i(t+T)$, repectively.

$$p(t) = v(t+T) \cdot i(t+T)$$

... Eq. (2)

Taking the integral of $p(t)$ over one period and dividing by T to find average power,

$$P(t) = \frac{1}{T} \cdot \int_{t_0}^{t_0+T} p(t)\, \mathrm{d}t$$

... Eq. (3)

Suppose in an AC circuit, the voltage is sinusoidal. Voltage is represented by

$$v(t) = V_m \cdot \cos\left(\omega \cdot t + \theta_v\right)$$

... Eq. (4)

where V_m is the maximum amplitude, ω is the frequency and θ_v is the phase shift. For a linear circuit at steady state, the current will be sinusoidal with the same frequency.

$$i(t) = I_m \cdot \cos\left(\omega \cdot t + \theta_i\right)$$

... Eq. (5)

Thus, instantaneous power is

$$p(t) = V_m \cdot I_m \cdot \cos(\omega \cdot t + \theta_v) \cdot \cos(\omega \cdot t + \theta_i)$$

... Eq. (6)

The product of two cosines is reduced by the trigonometric identity.

$$p(t) = V_m \cdot I_m \cdot \frac{1}{2} \cdot \left[\cos(\theta_v - \theta_i) + \cos(2 \cdot \omega \cdot t + \theta_v + \theta_i)\right]$$

... Eq. (7)

Average power for $t_0 = 0$ is then

$$P(t) = \frac{1}{T} \cdot \int_0^T \frac{V_m \cdot I_m}{2} \cdot \left[\cos(\theta_v - \theta_i) + \cos(2 \cdot \omega \cdot t + \theta_v + \theta_i)\right] \mathrm{d}t$$

... Eq (8)

Separating and removing the constants,

$$P(t) = \frac{V_m \cdot I_m \cdot \cos(\theta_v - \theta_i)}{2 \cdot T} \cdot \int_0^T \mathrm{d}t + \frac{V_m \cdot I_m}{2 \cdot T} \cdot \int_0^T \cos(2 \cdot \omega \cdot t + \theta_v + \theta_i) \, \mathrm{d}t$$

... Eq. (9)

The second integral will be zero because the average power over one period of a cosine function will be zero. Evaluating the integral reveals

$$P(t) = \frac{V_m \cdot I_m}{2} \cos(\theta_v - \theta_i)$$

... Eq. (10)

This is the average power that a circuit element will consume in one period.

Average Power Demo

restart;

Below, a basic circuit was created with a sin voltage amplitude of 12 V at 60 Hz and a resistance of 1 *Ω*. Sensors were connected to measure the voltage and current. The phase angle $\theta_v - \theta_i$ is 0 in this example because it consists of only reactive elements.

The graph displays the power of the circuit in terms of time.

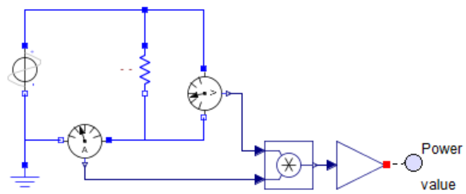


Figure 1.1: Simple R circuit with ammeter and voltmeter

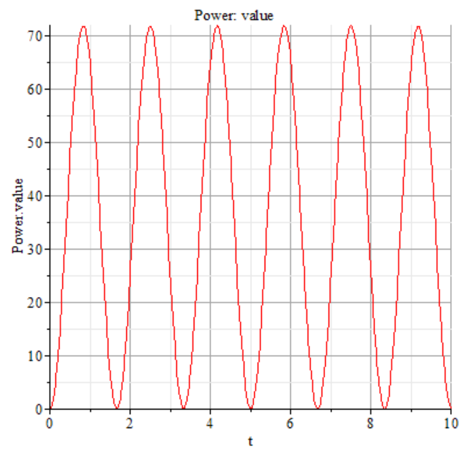


Figure 1.2: Power of a resistive circuit

The measured power was about 71.9 watts at it's peak. The angle between voltage and current is 0 so they both reach their peaks at the sime time, therefore we can use the power value shown here. To calculate this value, use the formula in Eq. (10). Knowing the resistance and max value of the voltage to be

$V_m := 12 [\![V]\!] :$

$R := 1 [\![\Omega]\!] :$

The current across the resistor will be

$$i_m := \frac{V_m}{R} = 12\ [\![A]\!]$$

So using Eq. (10) yields

$$P(t) := \frac{V_m \cdot I_m}{2} = 72\ [\![W]\!]$$

This is very close to the measured power from the model.

Complex Power

The element current and voltage can be represented in the frequency domain in the following forms.

$$I(\omega) = I_m \cdot e^{(j \cdot \theta_i)} = I_m \angle \theta_i \quad \text{and} \quad V(\omega) = V_m \cdot e^{(j \cdot \theta_v)} = V_m \angle \theta_v$$

... Eq. (11.1, 11.2)

This is equivalent to the time representations mentioned in Eq. (4) and Eq. (5) but is represented with repect to frequency.

Complex power S is defined as

$$S = \frac{V \cdot (I^*)}{2} = \frac{(V_m \angle \theta_v) \cdot (I_m \angle -\theta_i)}{2} = \frac{V_m \cdot I_m}{2} \cdot \angle(\theta_v - \theta_i)$$

... Eq. (12)

where I^* is the complex conjugate of I.

Apparent power is the magnitude of the complex power S.

$$|S| = \frac{V_m \cdot I_m}{2}$$

... Eq. (13)

Apparent power is simply the product of voltage and current, so it is measured in the unit VA, or volt-amperes.

Complex power has both a real and imaginary component to it. The real component is equivalent to **average** power P. This real power is the component of S which results in a net energy transfer in one direction. The imaginary component Q is due to elements like inductors and capacitors which 'lag' and 'lead' the current and store energy while operating. This is the **reactive** power of S. Reactive power is stored energy in the elements and returns to the source in each cycle.

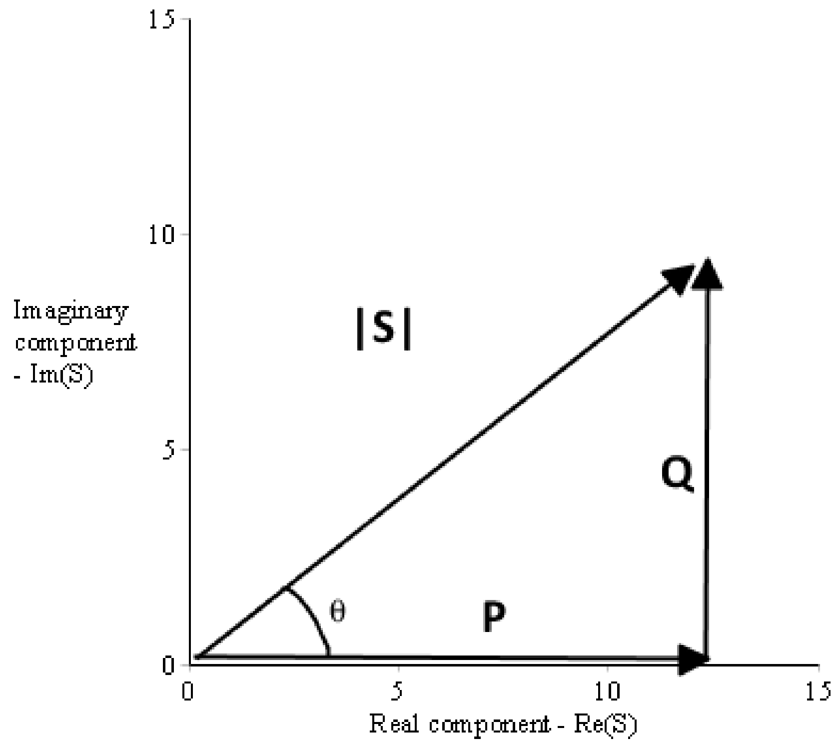


Figure 2: Complex power triangle

Complex power can be broken down into its real *P* and imaginary *Q* components, shown in the complex power triange in figure 1 above. The real component is the average power P in the circuit. Q is reactive power, the imaginary component of complex power. The phase angle θ is the difference between the sinusoidal voltage and current $\theta_v - \theta_i$ and is equivalent to the angle of the impedance *Z*.

In its components, S can be found as the sum of the vectors P and Q.

$$S = P + j \cdot Q$$

... Eq. (14)

Complex power converted to the time domain gives

$$S = \frac{V_m \cdot I_m}{2} \cdot \cos\left(\theta_v - \theta_i\right) + j \cdot \frac{V_m \cdot I_m}{2} \cdot \sin\left(\theta_v - \theta_i\right)$$

... Eq.(15)

Which implies, as expected, the components P and Q.

$$P = \frac{V_m \cdot I_m}{2} \cdot \cos\left(\theta_v - \theta_i\right) \quad \text{and} \quad Q = \frac{V_m \cdot I_m}{2} \cdot \sin\left(\theta_v - \theta_i\right)$$

... Eq. (16)

Impedance and Complex Power

The impedance of an element in an AC circuit is expressed as

$$Z(\omega) = \frac{V(\omega)}{I(\omega)} = \frac{V_m}{I_m}\frac{\angle\theta_v}{\angle\theta_i} = \frac{V_m}{I_m}\cdot\angle(\theta_v - \theta_i)$$

... Eq. (17)

Converting this form to rectangular,

$$Z(\omega) = \frac{V_m}{I_m}\cdot\cos(\theta_v - \theta_i) + j\cdot\frac{V_m}{I_m}\cdot\sin(\theta_v - \theta_i)$$

... Eq. (18)

Then the impedance can be given as complex components.

$$Z(\omega) = R + j\cdot X$$

... Eq. (19)

The form of Eq. (18) expresses impedance similar to the expression for complex power. By rearranging Eq. (15) and multiplying by $\frac{I_m}{I_m}$, this takes on the form

$$S = \left(\frac{I_m\cdot I_m}{2}\right)\cdot\frac{V_m}{I_m}\cdot\cos(\theta_v - \theta_i) + j\cdot\left(\frac{I_m\cdot I_m}{2}\right)\cdot\frac{V_m}{I_m}\cdot\sin(\theta_v - \theta_i)$$

... Eq. (20)

The real (Re) and imaginary (Im) components of the impedance are expressed in Eq. (20) now. Simplifying this expression gives

$$S = \left(\frac{I_m^{\,2}}{2}\right)\cdot\mathrm{Re}(Z) + j\cdot\left(\frac{I_m^{\,2}}{2}\right)\cdot\mathrm{Im}(Z)$$

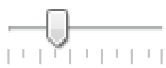
... Eq. (21)

Thus, complex power can be represented in terms of impedance.

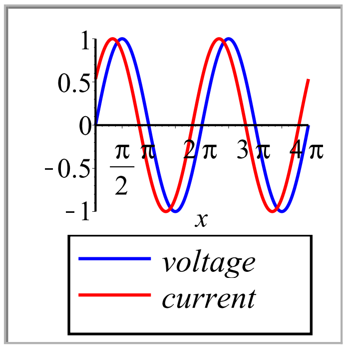
Interactive with Complex Power

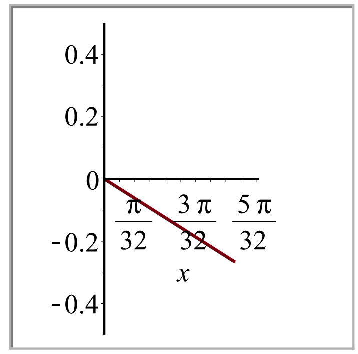
Manipulate the slider to change the phase angle of the current and observe the effects.

Current Phase Angle



$-\frac{\pi}{2}$ 0 $\frac{\pi}{2}$





Average Power = .423

Reactive Power = -.266

Apparent Power = .500

Table 1: Complex power demonstration

Which angle will provide the largest value of real power? Refer to the power triangle (figure 2) above for more information.

Power Factor

The power factor is a unitless value that describes the amount of 'real' power. It is a ratio between the average power and the apparent power.

$$pf = \frac{P}{|S|} = \frac{\left(\frac{V_m \cdot I_m}{2} \cdot \cos(\theta_v - \theta_i)\right)}{\left(\frac{V_m \cdot I_m}{2}\right)}$$

... Eq. (22)

The equation can be simplified into

$$pf = \cos\left(\theta_v - \theta_i\right)$$

... Eq. (#23)

The difference in angles $\theta_v - \theta_i$ is often reffered to as the **power factor angle** and is equivalent to the angle of impedance. However, an issue must be addressed. The cosine function is an even funtion such that,

$$\cos(x) = \cos(-x)$$

... Eq. (24)

which implies,

$$\cos\left(\theta_v - \theta_i\right) = \cos\left(\theta_i - \theta_v\right)$$

... Eq. (25)

So the power factor will always be positive. Thus a second parameter must specify whether the current is **leading** or **lagging** behind the voltage. If the current is lagging behind the voltage then $\theta_v > \theta_i$ and $\theta_v - \theta_i > 0$, so the angle will be positive. Alternatively, a leading current implies $\theta_v < \theta_i$. This means $\theta_v - \theta_i < 0$ and the angle will be negative.

A simple circuit with a capacitor, like the one shown below, would generate a leading current as shown in the figure 3.1 and 3.2.

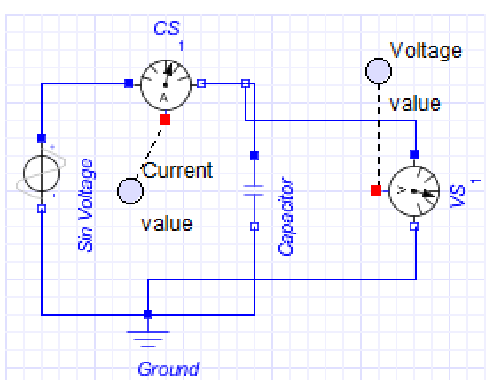


Figure 3.1: A sin signal attached to a capacitor with ammeter and voltmeter connected.

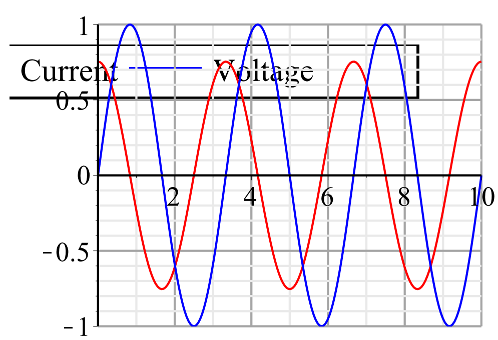


Figure 3.2: Current and voltage of circuit.

Here, the current is leading the voltage by a quarter of a cycle. The peak in the current signal happens before the peak in the voltage. Similarly, an inductor will lag the current by 1/4 of a period.

Maximum Power Transfer Theorum

Recall that in a purely resistive DC circuit the maximum power delivered to a load from a circuit will happen when the load resistance is equal to the Thevenin resistance of the source circuit. Examine this characteristic in a sinusoidal steady-state circuit.

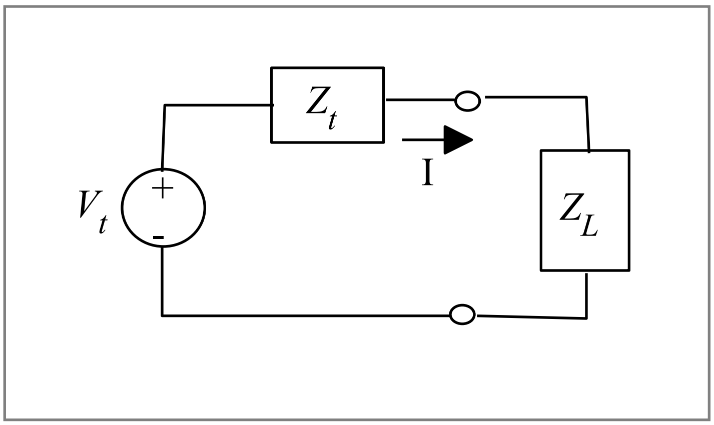


Figure 4: AC Thevenin circuit

Here, the impedances shown are

$$Z_t = R_t + j \cdot X_t \quad \text{and} \quad Z_L = R_L + j \cdot X_L$$

... Eq. (26.1, 26.2)

From Eq. (21), the average power applied on the load is

$$P = \frac{I_m^{\,2}}{2} \cdot R_L$$

... Eq. (27)

The current I is found to be

$$I = \frac{V_t}{Z_t + Z_L} = \frac{V_t}{(R_t + j \cdot X_t) + (R_L + j \cdot X_L)}$$

... Eq. (28)

Substituting current into the average power

$$P = \frac{1}{2} \cdot \frac{|V_t|^2 \cdot R_L}{(R_t + R_L)^2 + (X_t + X_L)^2}$$

... Eq. (29)

By setting $X_L = -X_t$ the reactance can be eliminated. Taking the derivative $\frac{dP}{dR_L}$ of this, the power will be maximum when the derivative equals zero. This will happen at $R_L = R_t$. So

$$P = \frac{|V_t|^2 \cdot R_t}{2(R_t + R_t)^2} = \frac{|V_t|^2}{8 \cdot R_t}$$

... Eq. (30)

The load impedance Z_L will become

$$Z_L = R_t - j \cdot X_t$$

... Eq. (31)

Therefore, the maximum power delivered to a circuit will be obtained as the load impedance Z_L is equal to the complex conjugate Z_t^* of the Thevenin equivalent impedance Z_t.

Examples with MapleSim

Example 1: Average Power

Example 2: Complex Power

References:

R. Dorf, J. Svoboda. "Introduction to Electric Circuits", 8th Edition. RRD Jefferson City, 2010, John Wiley and Sons, Inc.